Solutions 2 on 20 MAT 5190 M. ALVO

#3.23 a) $$\int_\alpha^\infty \frac{\beta \alpha^\beta}{x^{\beta+1}}\,dx = \beta \cdot \frac{\alpha^\beta x^{-\beta}}{\beta}\Big|_\alpha^\infty = 1$$

b) $$E[X] = \int_\alpha^\infty \beta \frac{\alpha^\beta}{x^\beta}\,dx = \frac{\alpha\beta}{\beta-1}, \qquad \beta > 1$$

$$E[X^2] = \int_\alpha^\beta \beta \frac{\alpha^\beta}{x^{\beta-1}}\,dx = \beta \frac{\alpha^2}{\beta-2} \qquad \beta > 2$$

(4)

$$\Rightarrow Var[X] = E[X^2] - E[X]^2 = \beta\frac{\alpha^2}{\beta-2} - \left(\frac{\alpha\beta}{\beta-1}\right)^2 = \frac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}, \beta > 2$$

c) If $\beta \le 2$, then $$E[X^2] = \int_\alpha^\infty \beta \frac{\alpha^\beta}{x^{\beta-1}}\,dx = \lim_{x\to\infty}\left[\frac{\beta\alpha^\beta}{x^{\beta-2}} + \frac{\beta\alpha^2}{\beta-2}\right]$$
and the limit does not exist

#5.15 a) $$\bar{X}_{n+1} = \frac{\sum_1^{n+1} X_i}{n+1} = \frac{X_{n+1} + \sum^n X_i}{n+1} = \frac{X_{n+1} + n\bar{X}_n}{n+1}$$

b) $$n S^2_{n+1} = \frac{n}{(n+1)-1} \cdot \sum_i^{n+1} \left(X_i - \bar{X}_{n+1}\right)^2$$

$$= \sum^{n+1}\left(X_i - \frac{X_{n+1} + n\bar{X}_n}{n+1}\right)^2$$

(5)

$$= \sum_1^{n+1}\left(X_i - \frac{X_{n+1}}{n+1} - \frac{n\bar{X}_n}{n+1}\right)^2$$

$$= \sum_1^{n+1}\left[\left(X_i - \bar{X}_n\right) - \left(\frac{X_{n+1}}{n+1} - \frac{\bar{X}_n}{n+1}\right)\right]^2$$

$$= \sum_1^{n+1}\left[\left(X_i - \bar{X}_n\right)^2 - 2\left(X_i - \bar{X}_n\right)\left(\frac{X_{n+1}}{n+1} - \frac{\bar{X}_n}{n+1}\right) + \left(\frac{X_{n+1}}{n+1} - \frac{\bar{X}_n}{n+1}\right)^2\right]$$

$$= \sum_{1}^{n+1}(X_i - \bar{X}_n)^2 - \frac{2}{n+1}(X_{n+1} - \bar{X}_n)^2 + (X_{n+1} - \bar{X}_n)^2 + \frac{n+1}{(n+1)^2}(X_{n+1} - \bar{X}_n)^2$$

$$= (n-1)S_n^2 + \frac{n}{n+1}(X_{n+1} - \bar{X}_n)^2$$

#5.22 $F_{Z^2}(z) = P((\min(X,Y))^2 \le z) = P(-\sqrt{z} \le \min(X,Y) \le \sqrt{z})$

$$= P(\min(X,Y) \le \sqrt{z}) - P(\min(X,Y) \le -\sqrt{z})$$

$$= P(\min(X,Y) > -\sqrt{z}) - P(\min(X,Y) > \sqrt{z})$$

(3) $= P(X > -\sqrt{z})P(Y > -\sqrt{z}) - P(X > \sqrt{z})P(Y > \sqrt{z})$ By indep.

$$= (1 - F_X(-\sqrt{z}))^2 - (1 - F_X(\sqrt{z}))^2 = 1 - 2F_X(-\sqrt{z})$$

Since $1 - F_X(\sqrt{z}) = F_X(-\sqrt{z})$

Diff. with respect to z

$$f_{Z^2}(z) = \frac{d}{dz}F_{Z^2}(z) = \frac{1}{\sqrt{z}}f_X(-\sqrt{z}) = \frac{1}{\sqrt{2\pi}} z^{-1/2} e^{-z/2}$$

which is the pdf of a χ_1^2 random variable.

#5.24 For $f_X(x) = \frac{1}{\theta}$, $F_X(x) = x/\theta$, $0 < x < \theta$

let $Y = X_{(n)}$, $Z = X_{(1)}$. Then from Theorem 5.4.6

$$f_{Z,Y}(z,y) = \frac{n(n-1)}{\theta^n}(y-z)^{n-2} \qquad 0 < z < y < \theta$$

let $W = Z/Y$, $V = Y$. Making the change of variable we

(3) calculate $|J| = v$ and hence

$$f_{W,V}(w,v) = \frac{n(n-1)}{\theta^n}(v - wv)^{n-2} v$$

$$= \frac{n(n-1)}{\theta^n}(1-w)^{n-2} v^{n-1} \qquad 0 < w < 1,\ 0 < v < \theta$$

$\Rightarrow$ W, V are independent since the joint density factors into a function of w times a function of v

#5.25 The joint density of $X_{(1)} \cdots X_{(n)}$ is

$$f(u_1, \dots, u_n) = n! \frac{a^n}{\theta^{an}} u_1^{a-1} \cdots u_n^{a-1} \qquad 0 < u_1 < \cdots < u_n < \theta$$

Make the transformation

(5)

$$Y_1 = X_{(1)}/X_{(2)}$$

$$\vdots$$

$$Y_{n-1} = X_{(n-1)}/X_{(n)}$$

$$Y_n = X_{(n)}$$

The Jacobian $|J| = y_2 y_3^2 \cdots y_n^{n-1}$

$$\therefore f(y_1 \cdots y_n) = \frac{n!\, a^n}{\theta^{an}} \left(\prod_1^n y_i\right)^{\alpha-1} \left(\prod_2^n y_i\right)^{\alpha-1} \cdots \left(y_n\right)^{\alpha-1} \left(y_2 y_3^2 \cdots y_n^{n-1}\right)$$

$$= \frac{n!\, a^n}{\theta^{an}} y_1^{\alpha-1} y_2^{2a-1} \cdots y_n^{na-1} \qquad 0 < y_i < 1, \ i = 1 \to n-1$$

$$0 < y_n < \theta$$

$\Rightarrow$ $Y_1 \cdots Y_n$ are independent

Now let $f_{Y_1}(y_1) = c_1 y_1^{\alpha-1}$ $\quad 0 < y_1 < 1$

$\Rightarrow$ $\quad c_1 = a$

Similarly if $f_{Y_i}(y_i) = c\, y_i^{ia-1}$ $\quad 0 < y_i < 1$

we get $\quad c = ia$

Also $\quad f_{Y_n}(y_n) = \frac{na}{\theta^{na}} y_n^{na-1}$ $\quad 0 < y_n < \theta$